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On the square of x^{-n}

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Abstract. The square of x^{-n} is evaluated, using Raju's definition of the pointwise product of distributions, and shown to differ from x^{-2n} by a finite multiple of the infinite (non-standard) distribution corresponding to the square of $\delta^{(n-1)}$. This result is used to derive identities establishing the finiteness of one-variable analogues of some propagator products in quantum electrodynamics. The one-variable identities may be applied to distributions f, g, concentrated on the null cone, $\lambda = 0$, by defining $f(\lambda) \cdot g(\lambda) = fg(\lambda)$ and regularising to include the vertex of the null cone. It is pointed out that this procedure would lead to a finite electron self-energy without restricting the domain of the S matrix and without introducing an infinite difference between the bare and renormalised mass. It is concluded that any polynomial in δ_- and its derivatives is finite and that unrestricted associativity holds for products of such polynomials.

1. Introduction

Mikusinski (1966) established the following identity for the square of the Dirac delta function:

$$\delta^2 - \frac{1}{\pi^2}\left(\frac{1}{x}\right)^2 = -\frac{1}{\pi^2}\frac{1}{x^2} \tag{1.1}$$

where the left-hand side is to be regarded as the distributional limit of $\delta_n^2 - \pi^{-2}(x^{-1} \otimes \delta_n)^2$, no meaning being assigned to the individual terms. δ_n denotes an appropriate δ-convergent sequence and $\otimes$ denotes convolution.

We observe that, with Mikusinski's definition, $(x^{-1})^2$ is *not* the same as x^{-2} and, in fact, differs from it by an infinite amount. This peculiarity arises because the *distribution* x^{-1}, corresponding to the principal value of x^{-1}, differs from the function x^{-1}. More interestingly, the state of affairs represented by (1.1) turns out to be of crucial importance for a rigorous resolution of the renormalisation problem (§ 2).

In addition, the identity

$$\delta \cdot x^{-1} = -\tfrac{1}{2}\delta' \tag{1.2}$$

has been proved by several authors including Gonzalez-Dominguez and Scarfiello (1955), Mikusinski (1966), Fisher (1971). The result is also true with the symmetric product defined by Raju (1982a, to be referred to as R), while both (1.1) and (1.2) follow if the product defined by Jones (1982) is symmetrised. Guttinger (1955) has derived a slightly different result. These two identities constitute the principal results, relevant to renormalisation, that have so far been obtained for products of distribution.

In this paper we establish a generalisation of the above identities using the symmetric product defined in R. Our results suggest that the propagator products arising in quantum electrodynamics do *not* involve either infinite or arbitrary constants.

There exist many different definitions of the product, but the definition in R has the following advantage. The usefulness of results like (1.1) and (1.2) is diminshed to the extent that the definition of the product involves arbitrariness. To restrict the arbitrariness possible in a definition, it was proposed in R that a definition, to be acceptable, should be applicable without restriction to both classical and quantum field theory. At present, no other definition of the product satisfies this criterion.

Thus, the definitions of Konig (1953, 1955), Guttinger (1955), Taylor (1960), Bogoliubov and Parasiuk (1959), Akheizer and Berestetskii (1965) and de Jager (1964) are obtained by imposing (implicitly or explicitly) special restrictions on the space of test functions. The products, so defined, may be extended to the entire space of test functions, using the Hahn–Banach theorem, only at the expense of uniqueness. That is, the net result of this procedure is to replace the usual infinities by finite but arbitrary constants. Therefore, these definitions of the product of distributions must be supplemented by phenomenology in quantum field theory, and cannot be used at all in classical field theory.

Although Mikusinski (1966), Fisher (1971, 1972, 1973), and Jones (1966, 1980, 1982) use different techniques, these definitions too are not always applicable. For example, the first two do not define δ^2, whereas the last yields $\delta^2=0$. So when δ^2 occurs in isolation† the above definitions may not be used.

On the other hand, the results obtained here and earlier applications to the calculation of transition probabilities and classical field theory (R, Raju 1982b) show that the definition in R does satisfy the above criterion. Naturally, one would prefer that definition of the product which is most generally applicable.

Below we illustrate the relevance of (1.1) and (1.2) to the renormalisation problem.

2. The problem of infinite self-energy

The second-order term of the S matrix corresponding to the electron self-energy is, with usual notation (Bogoliubov and Shirkov 1959, to be referred to as BS),

$$-\mathrm{i}{:}\bar{\Psi}(x)\underline{\Sigma}(x-y)\Psi(y){:} \tag{2.1}$$

where

$$\underline{\Sigma}(x)=-\mathrm{i}\,e^2\sum_n g^{nn}\gamma^n S^c(x)\gamma^n D_0^c(x) \tag{2.2}$$

involves the product of the propagators (Green functions) $S^c(x)$ and $D_0^c(x)$. Here g^{mn} is the metric tensor: $g^{00}=-g^{11}=-g^{22}=-g^{33}=1$, $g^{mn}=0$ if $m\neq n$. The Dirac matrices γ^n are defined by (BS, p 59)

$$\gamma^n\gamma^m+\gamma^m\gamma^n=2g^{mn}. \tag{2.3}$$

† This happens while calculating transition probabilities in quantum field theory (R), and in the study of gravitational screening (Raju 1982b) in classical field theory. In fact, according to the Einstein–Maxwell theory, a δ^2 energy density may be obtained formally by an electric dipole distribution on a hypersurface (Raju 1983).

The photon (scalar) and electron (spinor) propagators are respectively given by (BS, pp 651–2)

$$D_0^c(x) = D^c(x)|_{m=0} \tag{2.4}$$

$$S^c(x) = (\mathrm{i}\hat{\partial} + m)D^c(x) \qquad \hat{\partial} = \sum_{\mu=0}^{3} \gamma^\mu \partial_\mu. \tag{2.5}$$

To obtain explicit expressions for these propagators, we express the approximate form of $D^c(x)$, near the null cone, as follows:

$$D^c(x) = (1/2\pi)\delta_-(\lambda) - (m^2/8\pi)L \tag{2.6}$$

where

$$\delta_-(\lambda) = \tfrac{1}{2}(\delta(\lambda) - \mathrm{i}/\pi\lambda) \qquad L = \tfrac{1}{2}[\theta(\lambda) - (2\mathrm{i}/\pi)\log \tfrac{1}{2}m|\lambda|^{1/2}]$$

$$\lambda = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{2.7}$$

and θ is the Heaviside function.

If we naively assume the chain rule

$$\partial_\mu f(\lambda) = f'(\lambda)\partial_\mu \lambda \tag{2.8}$$

where f is a one-dimensional distribution and f' its derivative, then we have

$$\hat{\partial} f(\lambda) = 2f'(\lambda)\hat{x} \tag{2.9}$$

where $\hat{x}$ is defined as in (2.5). Observing that $L' = \delta_-$, we obtain

$$D_0^c(x) = (1/2\pi)\delta_-(\lambda)$$

$$S^c(x) = \frac{\mathrm{i}}{\pi}\delta'_-(\lambda)\hat{x} - \frac{\mathrm{i}m^2}{4\pi}\delta_-(\lambda)\hat{x} + \frac{m}{2\pi}\delta_-(\lambda) - \frac{m^3}{8\pi}L. \tag{2.10}$$

Using the well known properties of the Dirac matrices (BS, p 290, Bjorken and Drell 1964)

$$\sum_n g^{nn}\gamma^n \hat{a}\gamma^n = -2\hat{a} \qquad \sum_n g^{nn}\gamma^n\gamma^n = 4 \tag{2.11}$$

and substituting (2.10) in (2.2), we obtain the following expression for the 'singular' part of $\underline{\Sigma}(x)$:

$$(\mathrm{i}\pi^2/e^2)\underline{\sum}(x) = -\mathrm{i}\delta'_- \cdot \delta_- \hat{x} + m\delta_-^2 + \tfrac{1}{4}\mathrm{i}m^2\delta_-^2\hat{x} - \tfrac{1}{4}m^3 L \cdot \delta_- \tag{2.12}$$

the argument λ having been dropped for simplicity.

We wish to obtain a finite S matrix in such a manner that (a) the finiteness of the S matrix does not depend on the value of the phenomenological constant m, and (b) recourse to the fiction of a bare mass, infinitely different from the phenomenological mass, is not necessary. Clearly, a *necessary* condition for this to happen is that the following products of one-dimensional distributions should be finite:

$$A = \delta'_- \cdot \delta_- \qquad B = \delta_-^2 = \frac{1}{4}\left[\delta^2 - \frac{1}{\pi^2}\left(\frac{1}{x}\right)^2 - \frac{\mathrm{i}}{\pi}\delta \cdot \frac{1}{x} - \frac{\mathrm{i}}{\pi x} \cdot \delta\right]$$

$$C = L \cdot \delta_-. \tag{2.13}$$

If we use the symmetric product defined in R, then by the Leibniz rule (R, theorem 3)

$$\delta_- \odot \delta'_- = \tfrac{1}{2}(\delta_-^2)' \qquad \delta_- \odot L = \tfrac{1}{2}(L^2)' \tag{2.14}$$

and the major difficulty lies in proving the finiteness of δ_-^2. Evidently, this can be done provided the identities (1.1) and (1.2) are valid with the symmetric product.

Similarly, the photon self-energy diagram corresponds to the term (BS, pp 293–4)

$$-\mathrm{i}\sum_{m,n} :A_m(x)\Pi^{mn}(x-y)A_n(y): \tag{2.15}$$

where

$$\Pi^{mn}(x) = -\mathrm{i}\, e^2 \operatorname{Tr} \gamma^m S^c(x)\gamma^n S^c(-x) = -\mathrm{i}\, e^2 \operatorname{Tr} \gamma^m S^c(x)\gamma^n S^c(x) \tag{2.16}$$

since S^c is actually a function of λ, and λ is an even function of x. Proceeding in the manner indicated above, we see that the major difficulty lies in proving the finiteness of

$$(\delta'_-)^2 = \tfrac{1}{4}[(\delta')^2 - (1/\pi^2)(1/x^2)^2 + (2\mathrm{i}/\pi)\delta' \odot (1/x^2)]. \tag{2.17}$$

We reiterate that the above examples are illustrative and that the finiteness of δ_-^2, $(\delta'_-)^2$ etc, in the one-variable case, only provides necessary conditions for the finiteness of self energies. However, these necessary conditions are non-trivial since, for any one-variable distribution f, $f(\lambda)$ behaves (locally) like a one-variable distribution about any point of the null cone other than the vertex.

In fact, we could define the product of two distributions, concentrated on the null cone, by

$$f(\lambda)\cdot g(\lambda) = fg(\lambda) \tag{2.18}$$

the right-hand side being defined over the entire null cone by means of regularisation (Gel'fand and Shilov 1964). From the final formulae we obtain, it is clear that such a regularisation would certainly exist and be unique for products of the type $\delta_-^{(n)}\cdot\delta_-^{(n)}$. Thus, with an additional condition like (2.18) our results are just as well applicable to the four-dimensional case. A subsequent paper will present a more detailed account of the four-dimensional situation, based largely on the results proved here for the one-dimensional case.

3. Generalisation of Mikusinski's identity

D, S will denote the spaces of infinitely differentiable functions on the real line that are compactly supported and rapidly decreasing respectively. D', S' will denote the corresponding spaces of distributions and tempered distributions. $D(0, 1, 2, \ldots, n) = \{\phi \in D,\ \phi(0) = \phi'(0) = \ldots \phi^{(n)}(0) = 0\}\cdot \delta_n$ will denote a delta convergent sequence obtained by putting

$$\delta_n(x) = n\sigma(nx) \tag{3.1}$$

where σ is a symmetric, infinitely differentiable function with support contained in $[-1, 1]$, $\int \sigma(x)\,\mathrm{d}x = 1$ and $\sigma(0) \neq 0$.

For $f, g \in D'$,

$$f\cdot g = \lim_{n=\omega} {}^*(f_n g) \qquad f \odot g = \tfrac{1}{2}(f\cdot g + g\cdot f) \qquad f_n = f \otimes \delta_n \tag{3.2}$$

where ${}^*(f_n g)$ denotes the non-standard extension (Stroyan and Luxemburg 1976) of the sequence of distributions $f_n g$ and $\lim_{n=\omega}$ denotes the evaluation of the ωth term of this sequence for a fixed positive infinite integer ω. When $f \cdot g$ is finite, this operation corresponds to taking the distributional limit of $f_n g$. To simplify notation, the * will not be used in future.

The distributions x^{-n} are defined by

$$\langle x^{-1}, g\rangle = P\int x^{-1}g = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} x^{-1}g \qquad n!x^{-n-1} = (-1)^n (x^{-1})^{(n)} \tag{3.3}$$

where $f^{(n)}$ denotes the nth distribution derivative of f.

The first theorem generalises (1.1). To avoid dealing with subtle relationships between different notions of convergence, we shall present a straightforward though slightly long proof.

Theorem 1.

$$(\delta^{(n)})^2 - \frac{1}{\pi^2}\left(\frac{n!}{x^{n+1}}\right)^2 = -\frac{(n!)^2}{\pi^2}\frac{1}{x^{2n+2}} \qquad n = 0, 1, \ldots. \tag{3.4}$$

Before proving the theorem, we state and prove some lemmas which establish the nature of the distribution $(x^{-n})^2$.

Lemma 1.

$$\langle x^{-n}, x^n h\rangle = \langle 1, h\rangle \qquad \forall h \in D, n = 1, 2, 3, \ldots. \tag{3.5}$$

Proof. Clearly

$$\langle x^{-1}, xh\rangle = P\int x^{-1}(xh) = P\int h = \int h = \langle 1, h\rangle. \tag{3.6}$$

For $n \geq 1$

$$\begin{aligned}\langle x^{-n}, x^n h\rangle &= [(-1)^{n-1}/(n-1)!]\langle (x^{-1})^{(n-1)}, x^n h\rangle \\ &= [(n-1)!]^{-1}\langle x^{-1}, (x^n h)^{(n-1)}\rangle \\ &= \frac{1}{(n-1)!}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{n!}{(n-i)!}\langle x^{-1}, x^{n-i}h^{(n-i-1)}\rangle \end{aligned}\tag{3.7}$$

by Leibniz's rule. But, by (3.6), for $i \leq n-1$

$$\begin{aligned}\langle x^{-1}, x^{n-i}h^{(n-i-1)}\rangle &= \langle 1, x^{n-i-1}h^{(n-i-1)}\rangle = \langle x^{n-i-1}, h^{(n-i-1)}\rangle \\ &= (-1)^{n-i-1}\langle (x^{n-i-1})^{(n-i-1)}, h\rangle \\ &= (-1)^{n-i-1}(n-i-1)!\langle 1, h\rangle. \end{aligned}\tag{3.8}$$

Substituting in (3.7)

$$\langle x^{-n}, x^n h\rangle = \langle 1, h\rangle \sum_{i=0}^{n-1}(-1)^{n-i-1}\binom{n}{i} = \langle 1, h\rangle. \tag{3.9}$$

Lemma 2. For $n = 1, 2, 3, \ldots$

$$\langle (x^{-n})^2, h\rangle = \langle x^{-2n}, h\rangle \qquad \forall h \in D(0, 1, 2, \ldots, n-1). \tag{3.10}$$

Proof. For $h \in D$ $(0, 1, 2, \ldots, n-1)$, there exists a $g \in D$ such that $h = x^n g$ (Zemanian

1965). By lemma 1

$$\langle (x^{-n})^2, h\rangle = \lim_{m=\omega} \langle x_m^{-n} x^{-n}, x^n g\rangle = \lim_{m=\omega} \langle x^{-n}, x^n g x_m^{-n}\rangle$$

$$= \lim_{m=\omega} \langle x_m^{-n}, g\rangle$$

$$= \langle x^{-n}, g\rangle = [(n-1)!]^{-1} \langle x^{-1}, g^{(n-1)}\rangle. \tag{3.11}$$

On the other hand

$$\langle x^{-2n}, h\rangle = \langle x^{-2n}, x^n g\rangle = [(2n-1)!]^{-1} \langle x^{-1}, (x^n g)^{(2n-1)}\rangle$$

$$= \frac{1}{(2n-1)!} \sum_{i=0}^{n} \binom{2n-1}{i} \frac{n!}{(n-i)!} \langle x^{-1}, x^{n-i} g^{(2n-1-i)}\rangle. \tag{3.12}$$

If $i < n$, then

$$\langle x^{-1}, x^{n-i} g^{(2n-1-i)}\rangle = \langle x^{n-i-1}, g^{(2n-1-i)}\rangle$$

$$= (-1)^{n-i} \langle (x^{n-i-1})^{(n-i)}, g^{(n-1)}\rangle$$

$$= 0. \tag{3.13}$$

Thus, the only term that survives in the summation in (3.12) is the term corresponding to $i = n$, and this term yields the contribution

$$\frac{1}{(2n-1)!} \binom{2n-1}{n} n! \langle x^{-1}, g^{(n-1)}\rangle = \frac{1}{(n-1)!} \langle x^{-1}, g^{(n-1)}\rangle. \tag{3.14}$$

The result follows.

Corollary 1.

$$(x^{-n})^2 - x^{-2n} = \sum_{i=0}^{n-1} b_i^n \delta^{(i)}. \tag{3.15}$$

Proof. $(x^{-n})^2 - x^{-2n}$ vanishes on $D(0, 1, 2, \ldots, n-1)$ which is the null space of δ, $\delta', \ldots, \delta^{(n-1)}$. The result follows from a standard theorem (Rudin 1974).

Lemma 3. As $m \to \infty$, $x_m^{-n} \to x^{-n}$ uniformly on any set $A \subset \mathbb{R}$ such that 0 does not belong to the closure of A, $n = 1, 2, 3, \ldots$.

Proof. By definition,

$$x_m^{-n}(x) = \int_{-1/m}^{1/m} \frac{m\sigma(my)}{(x-y)^n} \mathrm{d}y \qquad |x| > \frac{1}{m}$$

$$= \int_{-1}^{1} \frac{\sigma(z)}{(x - z/m)^n} \mathrm{d}z \qquad |x| > \frac{1}{m} \tag{3.16}$$

by a simple change of variables ($my = z$).

Since 0 does not belong to the closure of A

$$k = \inf\{|x|, x \in A\} > 0. \tag{3.17}$$

If we choose m so large that $(mk)^{-1} \ll 1$, then for $x \in A$

$$\left(x-\frac{z}{m}\right)^{-n}=x^{-n}\sum_{r=0}^{\infty}\binom{n+r-1}{r}\left(\frac{z}{mx}\right)^{r}. \tag{3.18}$$

Since the power series is uniformly convergent for $|z| \leqslant 1$, so that $|z/mx| \leqslant 1/mk$, one may integrate term by term to obtain

$$x_m^{-n}(x)=x^{-n}\left[\int_{-1}^{1}\sigma(z)\,\mathrm{d}z+\sum_{r=1}^{\infty}\left(\frac{1}{mx}\right)^{r}\binom{n+r-1}{r}\int_{-1}^{1}z^{r}\sigma(z)\,\mathrm{d}z\right] \tag{3.19}$$

which may be rewritten as

$$x_m^{-n}(x)-x^{-n}=x^{-n}\sum_{r=1}^{\infty}(mx)^{-r}\binom{n+r-1}{r}\int_{-1}^{1}z^{r}\sigma(z)\,\mathrm{d}z \tag{3.20}$$

using $\int_{-1}^{1}\sigma(z)\,\mathrm{d}z=1$. To obtain a uniform bound on the right-hand side of (3.20), we observe that $|z^r| \leqslant 1$ for $|z| \leqslant 1$ and put $\int_{-1}^{1}|\sigma(z)|\mathrm{d}z=\beta$ so that, for $x \in A$,

$$\begin{aligned}|x_m^{-n}(x)-x^{-n}| &\leqslant \beta|x|^{-n}\sum_{r=1}^{\infty}\binom{n+r-1}{r}|mx|^{-r}\\ &\leqslant \beta k^{-n}\sum_{r=1}^{\infty}\binom{n+r-1}{r}(mk)^{-r}\\ &=\beta k^{-n}[(1-1/mk)^{-n}-1].\end{aligned} \tag{3.21}$$

The result follows by observing that $\lim_{m\to\infty}(1-1/mk)^{-n}=1$.

Lemma 4.

$$b_i^n=\lim_{m=\omega}\frac{(-1)^{n-1+i}}{i!}\binom{2n-2-i}{n-1}\mathrm{P}\int x^{-1}x_m^{-2n+i+1} \tag{3.22}$$

where b_i^n are defined by (3.15), for $0 \leqslant i \leqslant n-1$, $n=1, 2, 3, \ldots$.

Proof. We select functions $h_i \in D$ which behave like x^i in a neighbourhood B of zero. Then

$$\langle (x^{-n})^2-x^{-2n}, h_i\rangle=(-1)^i i!\, b_i^n. \tag{3.23}$$

Using the definitions of x^{-2n} and $(x^{-n})^2$, and assuming without loss of generality that B contains the interval $[-1, 1]$, we obtain

$$\begin{aligned}\langle x^{-2n}, h_i\rangle &=[(2n-1)!]^{-1}\langle x^{-1}, h_i^{(2n-1)}\rangle\\ &=\frac{1}{(2n-1)!}\left(\mathrm{P}\int_{-1}^{1}+\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)(x^{-1}h_i^{(2n-1)})\end{aligned} \tag{3.24}$$

$$\begin{aligned}\langle (x^{-n})^2, h_i\rangle &=\lim_{m=\omega}\frac{1}{(n-1)!}\langle x^{-1}, (x_m^{-n}\cdot h_i)^{(n-1)}\rangle\\ &=\lim_{m=\omega}\frac{1}{(n-1)!}\left(\mathrm{P}\int_{-1}^{1}+\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)[x^{-1}(x_m^{-n}h_i)^{(n-1)}].\end{aligned} \tag{3.25}$$

Since $h_i=x^i$ on $[-1, 1]$, and $i \leqslant n-1 < 2n-1$ for $n \geqslant 1$, $h_i^{(2n-1)}=0$ on $[-1, 1]$. So

$$\mathrm{P}\int_{-1}^{1}x^{-1}h_i^{(2n-1)}=0. \tag{3.26}$$

Before evaluating the remaining integrals in (3.24) and (3.25) we use the elementary formula for integration by parts

$$\int_a^b fg^{(n)} = \sum_{j=0}^{n-1} [(-1)^j f^{(j)} g^{(n-1-j)}]_a^b + (-1)^n \int_a^b f^{(n)} g \tag{3.27}$$

and define

$$c_{1i}^{+} = \frac{1}{(2n-1)!} \int_1^\infty x^{-1} h_i^{(2n-1)} - \int_1^\infty x^{-2n} h_i$$

$$c_{2i}^{+}(m) = \frac{1}{(n-1)!} \int_1^\infty x^{-1} (x_m^{-n} h_i)^{(n-1)} - \int_1^\infty x^{-n} x_m^{-n} h_i$$

$$c_{2i}^{+} = \lim_{m \to \infty} c_{2i}^{+}(m). \tag{3.28}$$

c_{1i}^{-} and c_{2i}^{-} are defined analogously, with the limits of integration being taken from $-\infty$ to -1. Since $x_m^{-n} \to x^{-n}$ uniformly, as $m \to \infty$, on $[1, \infty)$ and $(-\infty, -1]$, and h_i is bounded

$$\lim_{m \to \infty} \int_1^\infty x^{-n} x_m^{-n} h_i = \int_1^\infty x^{-2n} h_i. \tag{3.29}$$

Thus

$$(-1)^i i! \, b_i^n = c_{2i}^{+} + c_{2i}^{-} - c_{1i}^{+} - c_{1i}^{-} + \lim_{m=\omega} \frac{1}{(n-1)!} \mathrm{P} \int_{-1}^{1} x^{-1} (x_m^{-n} h_i)^{(n-1)}. \tag{3.30}$$

To avoid some laborious combinatorics in the evaluation of $c_{ji}^{\pm}$ we proceed as follows.

$$c_{1i}^{+} = \frac{1}{(2n-1)!} \sum_{j=0}^{2n-2} [(-1)^j (x^{-1})^{(j)} h_i^{(2n-2-j)}]_1^\infty$$

$$= -\frac{1}{(2n-1)!} \sum_{j=0}^{2n-2} (-1)^j (x^{-1})^{(j)} (x^i)^{(2n-2-j)}|_{x=1} \tag{3.31}$$

since h_i vanishes at ∞ and behaves like x^i in a neighbourhood of 1. It follows that

$$c_{1i}^{+} = \frac{1}{(2n-1)!} \sum_{j=0}^{2n-2} [(-1)^j (x^{-1})^{(j)} (x^i)^{(2n-2-j)}]_1^\infty. \tag{3.32}$$

The upper limit again contributes zero because, apart from a numerical factor, $(x^{-1})^{(j)} (x^i)^{(2n-2-j)}$ equals $x^{-2n+1+i}$ which vanishes at ∞. Using (3.27)

$$c_{1i}^{+} = \frac{1}{(2n-1)!} \int_1^\infty x^{-1} (x^i)^{(2n-1)} - \int_1^\infty x^{-2n} \cdot x^i = -\int_1^\infty x^{-2n+i} \tag{3.33}$$

since $i \leqslant n-1 < 2n-1$ so that $(x^i)^{(2n-1)}$ vanishes.

To evaluate c_{2i}^{+} similarly, we have only to recall the definition of x_m^{-k} and the standard rule for differentiating a convolution

$$(f \otimes g)^{(\alpha)} = f^{(\alpha)} \otimes g \qquad f \in S', g \in S, \alpha = 1, 2, 3, \ldots . \tag{3.34}$$

so that derivatives of x_m^{-k} may be calculated exactly like derivatives of x^{-k}. Moreover, lemma 3 applies in a neighbourhood of 1, so that

$$c_{2i}^{+} = \frac{1}{(n-1)!}\int_1^\infty x^{-1}(x^{-n+i})^{(n-1)} - \int_1^\infty x^{-2n+i}. \tag{3.35}$$

c_{1i}^{-} and c_{2i}^{-} are obtained similarly and it follows that

$$c_{2i}^{+} + c_{2i}^{-} - c_{1i}^{+} - c_{1i}^{-} = (-1)^{n-1-i}\binom{2n-2-i}{n-1}\left[\int_{-\infty}^{-1} + \int_1^\infty\right](x^{-2n+i}). \tag{3.36}$$

The additional factor of $(-1)^{-i}$ is permissible in the right-hand side of (3.36) which is 0 for i odd because the integrand is antisymmetric in that case.

To evaluate the remaining term in (3.30), we recall that $h_1 = x^i$ on $[-1, 1]$ and apply the Leibniz rule to obtain

$$\begin{aligned} &\frac{1}{(n-1)!}\lim_{m=\omega} \mathrm{P}\int_{-1}^{1} x^{-1}(x_m^{-n}h_i)^{(n-1)} \\ &\quad= \lim_{m=\omega}\sum_{j=0}^{i}(-1)^{n-1-j}\binom{i}{j}\binom{2n-2-j}{n-1}\mathrm{P}\int_{-1}^{1} x^{-1}x^{i-j}x_m^{-2n+1+j}. \end{aligned} \tag{3.37}$$

Amongst the integrals appearing in the above summation, the principal value needs to be taken only when $j = i$, and this term yields the contribution

$$(-1)^{n-1-i}\binom{2n-2-i}{n-1}\mathrm{P}\int_{-1}^{1} x^{-1}x_m^{-2n+1+i}. \tag{3.38}$$

To complete the proof of the lemma, it remains to show that $c_{3i} = 0$, where

$$c_{3i} = \lim_{m\to\infty}\sum_{j=0}^{i-1}(-1)^j\binom{i}{j}\binom{2n-2-j}{n-1}\int_{-1}^{1} x^{i-j-1}x_m^{-2n+1+j}. \tag{3.39}$$

The integrals in (3.39) are ordinary integrals since x_m^{-k} is an infinitely differentiable function as is x^{i-j-1} for $j \leq i-1$. To evaluate these integrals we observe that, apart from a numerical factor, $x_m^{-2n+j+1}$ is the $(i-j)$th derivative of $x_m^{-2n+i+1}$. Integrating by parts $(i-j)$ times and replacing x_m^{-k} by x^{-k} in the limit of the right-hand side as $m \to \infty$, we obtain

$$\begin{aligned} &\lim_{m\to\infty}\int_{-1}^{1} x^{i-j-1}(x_m^{-2n+i+1})^{(i-j)} \\ &\quad= \sum_{k=0}^{i-j-1}[(-1)^k(x^{i-j-1})^{(k)}(x^{-2n+i+1})^{(i-j-k-1)}]_{-1}^{1} \\ &\quad= -[\;]_1^\infty - [\;]_{-\infty}^{-1} \\ &\quad= -\left(\int_{-\infty}^{-1} + \int_1^\infty\right)[x^{i-j-1}(x^{-2n+i+1})^{(i-j)}]. \end{aligned} \tag{3.40}$$

It follows that

$$c_{3i} = -\left[\sum_{j=0}^{i-1}\binom{i}{j}\binom{2n-2-j}{n-1}(-1)^j\right]\left[\int_1^\infty x^{-2n+i} + \int_{-\infty}^{-1} x^{-2n+i}\right]. \tag{3.41}$$

For odd values of i the second factor on the right is zero, and for even values of i the first factor may be shown to be zero by applying the Leibniz rule to the ordinary function $(x^{-n} \cdot x^{i})^{(n-1)}$ and putting $x = 1$. Thus $c_{3i} = 0$ in all cases and the proof of the lemma is complete.

Proof of theorem 1. In view of

$$[\delta^{(n)}]^2 = (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} \delta_\omega^{(2n-j)}(0)\delta^{(j)} \tag{3.42}$$

and corollary 1 and lemma 4, (with n replaced by $n+1$), we have only to show that

$$[\pi^2/(2n-j)!]\delta_\omega^{(2n-j)}(0) = \mathrm{P}\int x^{-1} x_\omega^{-2n-1+j}. \tag{3.43}$$

If j is odd (3.43) holds, both sides vanishing on account of the assumed symmetry of σ in (3.1). If j is even, we proceed as follows.

We use the non-normalised Fourier transform

$$\tilde{f}(\alpha) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x \tag{3.44}$$

so that the Parseval equality reads

$$\langle f, \bar{\phi}\rangle = (1/2\pi)\langle \tilde{f}, \bar{\tilde{\phi}}\rangle \qquad f \in S', \qquad \phi \in S \tag{3.45}$$

where $\bar{\phi}$ denotes the complex conjugate of ϕ. Moreover (Gel'fand and Shilov 1964),

$$(x_m^{-n})^{\sim}(\alpha) = (x^{-n})^{\sim} \cdot \tilde{\delta}_m = \mathrm{i}^n \frac{\pi}{(n-1)!} \alpha^{n-1} \operatorname{sgn}\alpha \cdot \tilde{\delta}_m(\alpha) \tag{3.46}$$

since $\delta_m \in S$.

We wish to conclude that

$$\begin{aligned}
\langle x^{-1}, x_m^{-2n+j-1}\rangle &= (1/2\pi)\langle (x^{-1})^{\sim}, (x_m^{-2n+j-1})^{\bar{\sim}}\rangle \\
&= [\pi/2(2n-j)!] \int (-\mathrm{i})^{2n-j}\alpha^{2n-j}\bar{\tilde{\delta}}_m(\alpha)\,\mathrm{d}\alpha \\
&= [\pi/2(2n-j)!]\langle [\delta^{(2n-j)}]^{\sim}, \bar{\tilde{\delta}}_m\rangle \\
&= [\pi^2/(2n-j)!]\langle \delta^{(2n-j)}, \delta_m\rangle \\
&= [\pi^2/(2n-j)!]\delta_m^{(2n-j)}(0)
\end{aligned} \tag{3.47}$$

(where the last of (3.47) holds because j is even, so $(-1)^{2n-j} = 1$).

To justify this conclusion it is necessary to justify the use of Parseval's equality for x_m^{-k} which belongs to neither D nor S. But, for any k, x_m^{-k} is square integrable, by lemma 3, as soon as m is large enough. Moreover, the Fourier transform of x_m^{-k} does not change if x_m^{-k} is regarded as a square integrable function rather than a tempered distribution. So by Parseval's equality for square integrable functions

$$\begin{aligned}
\lim_{p\to\infty}\langle x_p^{-1}, x_m^{-2n+j-1}\rangle &= \lim_{p\to\infty} \frac{\pi}{2(2n-j)!} \int (-\mathrm{i})^{2n-j}\alpha^{2n-j}\tilde{\delta}_p(\alpha)\bar{\tilde{\delta}}_m(\alpha)\,\mathrm{d}\alpha \\
&= [\pi/2(2n-j)!] \int (-\mathrm{i})^{2n-j}\alpha^{2n-j}\bar{\tilde{\delta}}_m(\alpha)\,\mathrm{d}\alpha
\end{aligned} \tag{3.48}$$

since $\delta_m \in S$ so $\bar{\tilde{\delta}}_m \in S$ so $\alpha^{2n-j}\bar{\tilde{\delta}}_m(\alpha) \in S$.

Hence it would suffice to show that

$$\lim_{p\to\infty}\int x_p^{-1}x_m^{-k}=\mathrm{P}\int x^{-1}x_m^{-k}. \tag{3.49}$$

To prove this we select infinitely differentiable functions g_1 and g_2 so that

$$\begin{aligned} g_1(x)&=1, & x&<-2 & \qquad g_2(x)&=1, & x&>2\\ &=0, & x&\geqslant -1 & &=0, & x&\leqslant 1.\end{aligned} \tag{3.50}$$

If

$$\begin{aligned} g_3(x)&=1-g_1(x) & x&\leqslant 1\\ &=1-g_2(x) & x&\geqslant 1\end{aligned} \tag{3.51}$$

then g_3 is also infinitely differentiable and

$$\sum_{i=1}^{3} g_i(x)=1 \qquad \forall x\in\mathbb{R}$$

$$\sum_{i=1}^{3} g_i(x)\cdot x_m^{-k}(x)=x_m^{-k}(x) \qquad \forall x\in\mathbb{R} \tag{3.52}$$

$$\lim_{p\to\infty}\int x_p^{-1}(g_i x_m^{-k})=\mathrm{P}\int x^{-1}(g_i x_m^{-k}) \qquad i=1,2,3. \tag{3.53}$$

For $i=1, 2$, the above result is valid by lemma 3. For $i=3$ it is valid because $g_3x_m^{-k}\in D$. Summing over i we obtain (3.49) and the proof of the theorem is complete.

The next result generalises (1.2).

Theorem 2.

$$\delta^{(n-1)}\odot x^{-n}=\tfrac{1}{2}[(-1)^n(n-1)!/(2n-1)!]\delta^{(2n-1)}. \tag{3.54}$$

Proof. From equation (2.42) of R we have

$$\delta^{(n-1)}\cdot x^{-n}=\sum_{i=0}^{n-1}(-1)^i\frac{(2n-2-i)!}{i!(n-i-1)!}x_\omega^{-2n+1+i}(0)\delta^{(i)}+(-1)^n\frac{(n-1)!}{(2n-1)!}\delta^{(2n-1)}. \tag{3.55}$$

For the other product we have the corrected equation (R, Corrigendum)

$$x^{-n}\cdot\delta^{(n-1)}=\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{(2n-2-i)!}{(n-1)!}x_\omega^{-2n+1+i}(0)\delta^{(i)}. \tag{3.56}$$

The result follows by observing that $x_\omega^{-2n+1+i}(0)=0$ if i is even (because σ is symmetric), and if i is odd the coefficients of $\delta^{(i)}$ in (3.55) and (3.56) differ only in sign.

Corollary 2.

$$[\delta_\pm^{(n)}]^2=\mp\frac{\mathrm{i}}{2\pi}\frac{(n!)^2}{(2n+1)!}\delta_\pm^{(2n+1)}. \tag{3.57}$$

Here and elsewhere the top signs go with the top and bottom with bottom.

Proof. The proof follows by straightforward computation, using theorems 1 and 2, and will be omitted.

If we use the distributions

$$(x \pm \mathrm{i}\varepsilon)^{-n-1} = x^{-n-1} \mp (-1)^n \frac{\mathrm{i}\pi}{n!}\delta^{(n)} = \mp(-1)^n \frac{2\mathrm{i}\pi}{n!}\delta_{\pm}^{(n)} \qquad n = 0, 1, 2, \ldots \tag{3.58}$$

then (3.57) reads

$$[(x \pm \mathrm{i}\varepsilon)^{-n}]^2 = (x \pm \mathrm{i}\varepsilon)^{-2n}. \tag{3.59}$$

From (3.59) one may be inclined to guess that

$$(x \pm \mathrm{i}\varepsilon)^{-n} \odot (x \pm \mathrm{i}\varepsilon)^{-m} = (x \pm \mathrm{i}\varepsilon)^{-n-m}$$

and

$$[(x \pm \mathrm{i}\varepsilon)^{-n}]^m = (x \pm \mathrm{i}\varepsilon)^{-nm}.$$

However, with the product (3.2) it might happen that $(f^2)^2 \neq f \cdot f^3$. The next theorem shows that this pathology does not occur for symmetric products involving powers of δ_-.

Theorem 3.

$$\delta_-^p \odot \delta_-^{n+1-p} = \delta_-^{n+1} = (\mathrm{i}/2\pi)^n \delta_-^{(n)}/n! \qquad 1 \leqslant p \leqslant n, \qquad n = 1, 2, 3, \ldots. \tag{3.60}$$

Proof. Assume inductively that (3.60) holds. There is a basis for induction since for $n = 1$,

$$\delta_-^2 = \frac{1}{4}\left(-\frac{1}{\pi^2}\frac{1}{x^2} + \frac{\mathrm{i}}{\pi}\delta'\right) = \frac{\mathrm{i}}{2\pi}\delta_-' \tag{3.61}$$

from theorems 1 and 2.

By induction hypothesis applied to $\delta_-^{(j)}$ and $\delta_-^{(n-j)}$, and using $\delta_-^{(0)} = \delta_-$, we have for $0 \leqslant j \leqslant n$

$$\delta_-^{(j)} \odot \delta_-^{(n-j)} = (\mathrm{i}/2\pi)^{-n} j!(n-j)!\,\delta_-^{j+1} \odot \delta_-^{n+1-j}. \tag{3.62}$$

Another expression may be obtained for the left-hand side using the Leibniz rule. First suppose $n = 2k - 1$. Then

$$\delta_- \odot \delta_-^{(n)} = [\delta_- \odot \delta_-^{(n-1)}]' - \delta_-' \odot \delta_-^{(n-1)} \tag{3.631}$$

$$\delta_-' \odot \delta_-^{(n-1)} = [\delta_-' \odot \delta_-^{(n-2)}]' - \delta_-'' \odot \delta_-^{(n-2)} \tag{3.632}$$

$$\delta_-^{(j-1)} \odot \delta_-^{(n+1-j)} = [\delta_-^{(j-1)} \odot \delta_-^{(n-j)}]' - \delta_-^{(j)} \odot \delta_-^{(n-j)} \tag{3.63j}$$

$$\delta_-^{(k-1)} \odot \delta_-^{(k)} = [\delta_-^{(k-1)} \odot \delta_-^{(n-k)}]' - \delta_-^{(k)} \odot \delta_-^{(k-1)}. \tag{3.63k}$$

Successively substituting the right-hand side of each equation for the last term on the right-hand side of the preceding equation, and using the symmetry of the product, we have for $0 \leqslant j \leqslant n$

$$\delta_-^{(j)} \odot \delta_-^{(n-j)} = \sum_{r=j+1}^{k} (-1)^{r-j-1}[\delta_-^{(r-1)} \odot \delta_-^{(n-r)}]' + (-1)^{k-j}\delta_-^{(k-1)} \odot \delta_-^{(k)}. \tag{3.64}$$

But, by induction hypothesis (and $\delta_-^{(0)}=\delta_-$) for $1\leqslant r\leqslant n$

$$
\begin{aligned}
[\delta_-^{(r-1)} &\odot \delta_-^{(n-r)}]' \\
&= [(\mathrm{i}/2\pi)^{-r+1-n+r}(r-1)!\,(n-r)!\,\delta_-^{r}\odot\delta_-^{n+1-r}]' \\
&= [(\mathrm{i}/2\pi)^{1-n}(r-1)!\,(n-r)!\,\delta_-^{n+1}]' \\
&= \left[\left(\frac{\mathrm{i}}{2\pi}\right)^{1-n}\cdot\left(\frac{\mathrm{i}}{2\pi}\right)^{n}\frac{(r-1)!\,(n-r)!}{n!}\,\delta_-^{(n)}\right]' \\
&= (\mathrm{i}/2\pi)[(r-1)!\,(n-r)!/n!]\delta_-^{(n+1)}. \qquad (3.65)
\end{aligned}
$$

Moreover, $(3.63k)$ can be rewritten

$$\delta_-^{(k-1)}\odot\delta_-^{(k)}=\tfrac{1}{2}[\delta_-^{(k-1)}\odot\delta_-^{(n-k)}]'=\tfrac{1}{2}(\mathrm{i}/2\pi)\{[(k-1)!]^2/n!\}\delta_-^{(n+1)}. \qquad (3.66)$$

Hence, for $0\leqslant j\leqslant n$

$$\delta_-^{(j)}\odot\delta_-^{(n-j)}=\frac{\mathrm{i}}{2\pi}\,\frac{\delta_-^{(n+1)}}{n!}\left(\sum_{r=j+1}^{k}(-1)^{r-j-1}(r-1)!\,(n-r)!+\tfrac{1}{2}(-1)^{k-j}[(k-1)!]^2\right). \qquad (3.67)$$

Comparing (3.67) with (3.62) we see that, to prove the theorem for $n=2k-1$, it suffices to establish

$$\frac{j!\,(n-j)!}{(n+1)!}=\frac{1}{n!}\sum_{r=j+1}^{k}(-1)^{r-j-1}(r-1)!\,(n-r)!+\tfrac{1}{2}(-1)^{k-j}\frac{(k-1)!\,(k-1)!}{n!}. \qquad (3.68)$$

To prove (3.68) we write

$$\frac{(r-1)!\,(n-r)!}{n!}=\frac{\Gamma(r)\Gamma(2k-r)}{\Gamma(2k)}=\int_0^1 x^{r-1}(1-x)^{2k-r-1} \qquad (3.69)$$

so that, after interchanging summation and integration, (3.68) can be rewritten

$$\frac{\Gamma(j+1)\Gamma(2k-j)}{\Gamma(2k+1)}=\int_0^1\sum_{r=j+1}^{k}(-1)^{r-j-1}x^{r-1}(1-x)^{n-r}+\tfrac{1}{2}(-1)^{k-j}\frac{\Gamma(k)\Gamma(k)}{\Gamma(2k)}. \qquad (3.70)$$

The first term on the right of (3.70) can be simplified:

$$
\begin{aligned}
\sum_{r=j+1}^{k}&(-1)^{r-j-1}x^{r-1}(1-x)^{n-r} \\
&=(-1)^j(1-x)^n(-x)^{-1}\sum_{r=j+1}^{k}(-x)^r(1-x)^{-r} \\
&=(-1)^j(1-x)^n(-x)^{-1}\sum_{l=0}^{k-j-1}(-x)^{l+j+1}(1-x)^{-l-j-1} \\
&=(-1)^j(1-x)^{n-j-1}(-x)^j\sum_{l=0}^{k-j-1}(-x)^l(1-x)^{-l} \\
&=(-1)^j(1-x)^{n-j-1}(-x)^j\{1-[-x/(1-x)]^{k-j}\}/\{1-[-x/(1-x)]\} \\
&=(-1)^j\frac{(1-x)^{n-j}(-x)^j}{(1-x)^{k-j}}[(1-x)^{k-j}-(-x)^{k-j}] \\
&=(1-x)^{2k-j-1}x^j=(-1)^{k-j}(1-x)^{k-1}x^k \qquad (3.71)
\end{aligned}
$$

using $n = 2k - 1$. Integrating, using (3.69), we see that (3.70) reduces to

$$\frac{\Gamma(j+1)\Gamma(2k-j)}{\Gamma(2k+1)} = \frac{\Gamma(j+1)\Gamma(2k-j)}{\Gamma(2k+1)} - (-1)^{k-j}\frac{\Gamma(k)\Gamma(k+1)}{\Gamma(2k+1)} + \tfrac{1}{2}(-1)^{k-j}\frac{\Gamma(k)\Gamma(k)}{\Gamma(2k)} \tag{3.72}$$

which holds since $\Gamma(k+1) = k\Gamma(k)$ and $\Gamma(2k+1) = 2k\Gamma(2k)$.

When $n = 2k$, (3.63k) must be modified to read

$$\delta_-^{(k-1)} \odot \delta_-^{(n+1-k)} = [\delta_-^{(k-1)} \odot \delta_-^{(n-k)}]' - \delta_-^{(k)} \odot \delta_-^{(k)} \tag{3.63k'}$$

so that in place of (3.64) we now have

$$\delta_-^{(j)} \odot \delta_-^{(n-j)} = \sum_{r=j+1}^{k} (-1)^{r-j-1}[\delta_-^{(r-1)} \odot \delta_-^{(n-r)}]' + (-1)^{k-j}\delta_-^{(k)} \odot \delta_-^{(k)}. \tag{3.73}$$

Moreover, by corollary 2

$$[\delta_-^{(k)}]^2 = (\mathrm{i}/2\pi)[(k!)^2/(n+1)!]\delta_-^{(n+1)}. \tag{3.74}$$

Thus, to prove the theorem, it suffices to establish

$$\frac{j!(n-j)!}{(n+1)!} = \frac{1}{n!}\sum_{r=j+1}^{k} (-1)^{r-j-1}(r-1)!(n-r)! + (-1)^{k-j}\frac{k!k!}{(n+1)!}. \tag{3.75}$$

(3.75) may be proved along the same lines as (3.68), and this completes the proof of the theorem.

An analogous result is valid for δ_+.

Theorem 4.

$$\delta_+^j \odot \delta_+^{n+1-j} = \delta_+^{n+1} = (-\mathrm{i}/2\pi)^n \delta_+^{(n)}/n!.$$

Proof. The proof is similar to that of theorem 3 and will be omitted.

The conclusions below are immediate consequences of the above theorems.

4. Conclusions

With the symmetric product, any polynomial in δ_- and its derivatives is finite and unrestricted associativity holds for symmetric products of such polynomials. The result is also true for δ_+ separately, though the product $\delta_- \cdot \delta_+$ is infinite.

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